CENTRO-AFFINE NORMAL FLOWS ON CURVES: HARNACK'S ESTIMATE AND ANCIENT SOLUTIONS

MOHAMMAD N. IVAKI

ABSTRACT. We obtain Harnack's estimate for the planar p centro-affine normal flows on curves and we classify compact, origin-symmetric, ancient solutions to this family of flows if $1 \leq p < 4$. In particular, we classify origin-symmetric, compact ancient solutions to the planar affine normal flow.

1. INTRODUCTION

The setting for this paper is the two dimensional Euclidean space, $\mathbb{R}^2$. A compact convex subset of $\mathbb{R}^2$ with non-empty interior is called a *convex body*. We denote the set of smooth, origin-symmetric, strictly convex bodies in $\mathbb{R}^2$ by $\mathcal{K}_{sym}$.

We consider a class of extensions of the affine normal flow in centro-affine differential geometry, namely the p centro-affine normal flow (in short, the p-flow) introduced by Stancu [16], and we investigate its compact, origin-symmetric ancient solutions in $\mathbb{R}^2$. The p centro-affine normal flows are natural generalizations of the affine normal flow in the context of the p affine surface area. The initial purpose of defining p centro-affine normal flow was finding new global centro-affine invariants of smooth convex bodies in which a certain class of existing invariants arose naturally. Only the short time existence to the p-flow was then needed. Moreover, several interesting isoperimetric type inequalities were obtained via short time existence of the flow [16], and the p-flow approach led to a geometric interpretation of the L_ϕ surface area recently introduced by Ludwig and Reitzner [12]. Later on, the long time behavior of the planar flow was studied by the author in [8] using tools of affine differential geometry. It was proved there that the area preserving p-flow with $p \geq 1$ evolves any convex body in $\mathcal{K}_{sym}$ to the unit disk in Hausdorff distance, modulo $SL(2)$. A further application of the techniques developed in [8] to the L_{-2} Minkowski problem and the stability of the p-affine isoperimetric inequality were given in [9] and [10], respectively. It essential to say, the term *centro* in centro-affine differential geometry emphasizes that, contrary to affine differential geometry or classical differential geometry, Euclidean translations of an object in the ambient space are not allowed. Therefore, our study of the p centro-affine normal flows has been restricted to the class origin-symmetric convex bodies.

Let K be an origin-symmetric, strictly convex body, smoothly embedded in $\mathbb{R}^2$. Let

$$x_K : \mathbb{S}^1 \to \mathbb{R}^2,$$

be the Gauss parametrization of ∂K, the boundary of $K \in \mathcal{K}_{sym}$, where the origin of the plane is chosen to coincide with the center of symmetry of the body. The support function of ∂K is defined by

$$s_{\partial K}(z) := \langle x_K(z), z \rangle,$$

for each $z \in \mathbb{S}^1$. We denote the curvature of ∂K by κ and, furthermore, the radius of curvature of the curve ∂K by $\mathfrak{r}$, both as functions on the unit circle. The latter is related to the support

Date: November 13, 2012.

Key words and phrases. geometric evolution equations; centro-affine normal flows; Harnack's estimate; affine differential geometry; affine support function; dual convex body; p-affine isoperimetric inequality; Blaschke-Santaló inequality; ancient solutions.

function by

$$\mathfrak{r}[s](z) := \frac{\partial^2}{\partial\theta^2}s(z) + s(z),$$

where θ is the angle parameter on $\mathbb{S}^1$. Let $K_0 \in \mathcal{K}_{sym}$. We consider a family $\{K_t\}_t \in \mathcal{K}_{sym}$, and their associated smooth embeddings $x : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$, which are evolving according to the p centro-affine normal flow, namely,

$$\partial_t x := -\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}-\frac{1}{3}} \kappa^{\frac{1}{3}}\, z,\ x(\cdot,0) = x_{K_0}(\cdot),\ x(\cdot,t) = x_{K_t}(\cdot) \tag{1.1}$$

for a fixed $p \geq 1$.

It is this flow that we propose classifying the origin-symmetric, ancient solutions of it (see also the evolution equation (3.1) for an equivalent definition of the p-flow). The case $p = 1$, the well-known affine normal flow, was already addressed by Sapiro and Tannenbaum [15] and Andrews [1, 3]. Andrews, investigated the affine normal flow of compact, convex hypersurfaces in any dimension and showed that the volume preserving flow evolves any convex initial bounded open set exponentially fast, in the C^∞ topology, to an ellipsoid. In another direction, interesting results for the affine normal flow have been obtained in [11] by Loftin and Tsui regarding ancient solutions, and existence and regularity of solutions on non-compact strictly convex hypersurfaces. Ancient solutions are solutions that exist on a time interval $(-\infty, T)$, for a positive finite time T. Complete classifications has been done for the curve shortening flow [5] and for the Ricci flow on surfaces [6] by P. Daskalopoulos, R. Hamilton and N. Sesum. It has been shown in [6] that an ancient solution of the Ricci flow on $\mathbb{S}^2$ must be either round sphere or the King-Rosenau sausage model. See an exposition by B. Chow, [4], on a formula of Daskalopoulos, Hamilton and Sesum used in [6]. We classify compact, origin-symmetric, ancient solutions to the p centro-affine flow for $p \in [1, 4)$. This in particular includes the important problem of classification of compact, origin-symmetric, ancient solutions to the affine normal flow in two dimensions. Pertaining to the affine normal flow, a classification has been done in all dimensions except for in dimension two [11]. It was shown there that the only compact, ancient solutions to the affine normal flow in $\mathbb{R}^n$, $n \geq 3$, are ellipsoids.

We prove the following Proposition and Theorem:

Proposition (Harnack's estimate)**.** *Let $p \geq 1$. Along the flow (1.1) we have*

$$\partial_t\left(s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}t^{\frac{p}{2p+2}}\right) \geq 0,$$

on $\mathbb{S}^1 \times (0,T)$.

Theorem (Ancient solutions)**.** *Let $1 \leq p < 4$. The only compact, origin-symmetric, ancient solutions to the p-flow are homothetic ellipses.*

In the next section we obtain the Harnack's estimate for the p-flow. In the third section as an application, we classify compact, ancient solutions to this family of flows in the class $\mathcal{K}_{sym}$, if $1 \leq p < 4$. In particular, we classify compact, origin-symmetric, ancient solutions to the planar affine normal flow. Classification of ancient solutions in $\mathcal{K}_{sym}$ to the planar p-flow for $p \geq 4$ and classification of ancient solutions in $\mathcal{K}_{sym}$ to the p-flow in $\mathbb{R}^n$, for $n \geq 3$ and $p \geq 1$, is of great interest. In the class of convex bodies that are not origins-symmetric, the classification problem remains both open and important.

2. Harnack's estimate

In this section we follow [2] to obtain the Harnack's estimate.

Proof. For simplicity we set $\alpha=-\frac{p}{p+2}$. To prove the Proposition, using parabolic maximum principle we prove that the quantity defined by

$$\mathcal{R}:=t\mathcal{P}-\frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha} \tag{2.1}$$

remains negative as long as the flow exists. Here $\mathcal{P}$ is defined as follows

$$\mathcal{P}:=\partial_t\left(-s^{1+3\alpha}\mathfrak{r}^{\alpha}\right).$$

Lemma 2.1. [8] *Along the p-flow, (1.1), we have*

- $\partial_t s=-s^{1+3\alpha}\mathfrak{r}^{\alpha}$,
- $\partial_t\mathfrak{r}=-\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta}+s^{1+3\alpha}\mathfrak{r}^{\alpha}\right]$.

Using the evolution equations of s and $\mathfrak{r}$ we find

$$\begin{aligned}\mathcal{P}&=(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha}+\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta}+s^{1+3\alpha}\mathfrak{r}^{\alpha}\right]\\&:=(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha}+\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{Q}.\end{aligned} \tag{2.2}$$

Lemma 2.2. *We have the following evolution equation for $\mathcal{P}$ as long as the flow exists.*

$$\begin{aligned}\partial_t\mathcal{P}=&-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right]+\left[(3\alpha+1)(3\alpha+2)-\frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}\\&+\left[-3(3\alpha+1)+\frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P}-\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}}.\end{aligned}$$

Proof. We repeatedly use the evolution equation of s and $\mathfrak{r}$.

$$\begin{aligned}&\partial_t\mathcal{P}\\&=-(1+3\alpha)(1+6\alpha)s^{1+9\alpha}\mathfrak{r}^{3\alpha}-2\alpha(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha-1}\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta}+s^{1+3\alpha}\mathfrak{r}^{\alpha}\right]\\&-\alpha(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha-1}\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta}+s^{1+3\alpha}\mathfrak{r}^{\alpha}\right]\\&-\alpha(\alpha-1)s^{1+3\alpha}\mathfrak{r}^{\alpha-2}\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta}+s^{1+3\alpha}\mathfrak{r}^{\alpha}\right]^2-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right]\\&=-(1+3\alpha)(1+6\alpha)s^{1+9\alpha}\mathfrak{r}^{3\alpha}-3\alpha(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha-1}\mathcal{Q}\\&-\alpha(\alpha-1)s^{1+3\alpha}\mathfrak{r}^{\alpha-2}\mathcal{Q}^2-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right].\end{aligned}$$

By the definition of $\mathcal{Q}$, (2.2):

$$\mathcal{Q}^2=\frac{\mathcal{P}^2}{\alpha^2s^{2+6\alpha}\mathfrak{r}^{2\alpha-2}}-\frac{2(3\alpha+1)}{\alpha^2}\frac{\mathcal{P}\mathfrak{r}^2}{s}+\frac{(3\alpha+1)^2}{\alpha^2}s^{6\alpha}\mathfrak{r}^{2\alpha+2}$$

and

$$\mathcal{Q}=\frac{\mathcal{P}-(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha}}{\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}}.$$

Substituting these expressions into the evolution equation of $\mathcal{P}$ we find that

$$\begin{aligned}\partial_t\mathcal{P}=&-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right]+\left[(3\alpha+1)(3\alpha+2)-\frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}\\&-3(3\alpha+1)s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P}-\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}}+\frac{2(\alpha-1)(3\alpha+1)}{\alpha}s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P}.\end{aligned}$$

This completes the proof of Lemma 2.2. ∎

We now proceed to find the evolution equation of $\mathcal{R}$ which is defined by (2.1). First note that

$$-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta}=-t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P}_{\theta\theta}+\frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta}.$$

Therefore, by Lemma 2.2 and identity (2.2)

$$\begin{aligned}
&\partial_t\mathcal{R}\\
&= -t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right]+t\left[(3\alpha+1)(3\alpha+2)-\frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}\\
&+t\left[-3(3\alpha+1)+\frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P}-t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}}+\mathcal{P}+\frac{\alpha}{\alpha-1}\mathcal{P}\\
&-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta}+t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P}_{\theta\theta}-\frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta}\\
&+\frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)-\frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)\\
&+\frac{\alpha(3\alpha+1)}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+6\alpha}\mathfrak{r}^{2\alpha}\right)-\frac{\alpha(3\alpha+1)}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+6\alpha}\mathfrak{r}^{2\alpha}\right)\\
&= -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta}+t\left[(3\alpha+1)(3\alpha+2)-\frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}\\
&+t\left[-3(3\alpha+1)+\frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P}-t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}}+\mathcal{P}\\
&+\frac{\alpha}{\alpha-1}\mathcal{P}-\frac{\alpha}{\alpha-1}\mathcal{P}-t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P}+\frac{\alpha^2}{\alpha-1}s^{2+6\alpha}\mathfrak{r}^{2\alpha-1}+\frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1}\\
&= -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta}+t\left[(3\alpha+1)(3\alpha+2)-\frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}\\
&+t\left[-3(3\alpha+1)+\frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P}-t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}}+\mathcal{P}\\
&-t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P}+\frac{\alpha^2}{\alpha-1}s^{2+6\alpha}\mathfrak{r}^{2\alpha-1}+\frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1}.
\end{aligned}$$

To make the last computation useful, in the last expression, using the definition of $\mathcal{R}$, identity (2.1), we replace $t\mathcal{P}$ by $\mathcal{R}+\frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha}$. Therefore, at the point where the maximum of $\mathcal{R}$ is achieved we have

$$\begin{aligned}
&\partial_t\mathcal{R}\\
&\le \mathcal{R}\left[-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}-\frac{\alpha-1}{\alpha}\frac{\mathcal{P}}{s^{1+3\alpha}\mathfrak{r}^{\alpha}}+\left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha}-3(3\alpha+1)\right]s^{3\alpha}\mathfrak{r}^{\alpha}\right]\\
&+\frac{\alpha}{\alpha-1}\left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha}-3(3\alpha+1)\right]s^{2+6\alpha}\mathfrak{r}^{2\alpha}+\frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1}\\
&+t\left[(3\alpha+1)(3\alpha+2)-\frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}\\
&\le \mathcal{R}\left[-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}-\frac{\alpha-1}{\alpha}\frac{\mathcal{P}}{s^{1+3\alpha}\mathfrak{r}^{\alpha}}+\left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha}-3(3\alpha+1)\right]s^{3\alpha}\mathfrak{r}^{\alpha}\right].
\end{aligned}$$

To obtain the last inequality we used the fact that terms on the second and third line are negative for $p\geq 1$. Hence by the parabolic maximum principle we have $\mathcal{R}=t\mathcal{P}-\frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha}\leq 0$. Since at the time zero we have $\mathcal{R}\leq 0$. Negativity of $\mathcal{R}$ is equivalent to

$$\partial_t \ln\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)\geq \frac{\alpha}{1-\alpha}\frac{1}{t},$$

for $t>0$. From this we conclude that

$$\partial_t\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}t^{\frac{\alpha}{\alpha-1}}\right)\geq 0$$

for $t>0$. The proof of the main Proposition is complete. $\square$

By the Harnack's estimate every solution to the flow (1.1) satisfies

$$\partial_t\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)+\frac{p}{2t(p+1)}\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)\geq 0. \tag{2.3}$$

Therefore, we have the following Corollary.

Corollary 2.3. *Every ancient solution to the flow (1.1) satisfies*

$$\partial_t\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)\geq 0.$$

Proof. We let the flow starts from a fixed time $t_0<0$. Then, the inequality (2.3) becomes

$$\partial_t\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)+\frac{p}{2(t-t_0)(p+1)}\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)\geq 0.$$

Now letting t_0 goes to $-\infty$ proves the claim. $\square$

Corollary 2.4. *Every ancient solution to the flow (1.1) satisfies:*

$$\partial_t\left(\frac{\kappa^{\frac{1}{3}}}{s}\right)\geq 0.$$

Proof. The support function, s, is decreasing, on the time interval $(-\infty,T)$. The claim now follows from the previous Corollary. $\square$

3. Affine differential setting

We will now recall several definitions from affine differential geometry. Let $\gamma:\mathbb{S}^1\to\mathbb{R}^2$ be an embedded strictly convex curve with the curve parameter θ. Define $\mathfrak{g}(\theta):=[\gamma_\theta,\gamma_{\theta\theta}]^{1/3}$, where, for two vectors u,v in $\mathbb{R}^2$, $[u,v]$ denotes the determinant of the matrix with rows u and v. The affine arc-length is then given by

$$\mathfrak{s}(\theta):=\int_0^\theta\mathfrak{g}(\xi)d\xi.$$

Furthermore, the affine tangent vector $\mathfrak{t}$ and the affine normal vector $\mathfrak{n}$ are defined in this order, as follows:

$$\mathfrak{t}:=\gamma_{\mathfrak{s}},\ \ \mathfrak{n}:=\gamma_{\mathfrak{s}\mathfrak{s}}.$$

In the affine coordinate $\mathfrak{s}$, the following relation holds:

$$[\gamma_{\mathfrak{s}},\gamma_{\mathfrak{s}\mathfrak{s}}]=1.$$

Moreover, it can be easily verified that $\dfrac{s}{\kappa^{1/3}}=\dfrac{[\gamma,\gamma_\theta]}{[\gamma_\theta,\gamma_{\theta\theta}]^{1/3}}=\dfrac{[\gamma,\gamma_{\mathfrak{s}}]}{[\gamma_{\mathfrak{s}},\gamma_{\mathfrak{s}\mathfrak{s}}]^{1/3}}$. Since $[\gamma_{\mathfrak{s}},\gamma_{\mathfrak{s}\mathfrak{s}}]=1$, we conclude that $\dfrac{s}{\kappa^{1/3}}=[\gamma,\gamma_{\mathfrak{s}}]$. The quantity $\sigma:=\dfrac{s}{\kappa^{1/3}}$ is called the affine support function. It plays an important role in our argument.

Let K be a smooth convex body having origin in its interior. We can write the area of K, denoted by $A(K)$, in terms of affine invariant quantities:

$$A(K)=\frac{1}{2}\int_{\partial K}\sigma d\mathfrak{s}.$$

Furthermore, the p-affine perimeter of K, denoted by $\Omega_p(K)$, is defined by

$$\Omega_p(K) := \int_{\partial K} \sigma^{1-\frac{3p}{p+2}} d\mathfrak{s},$$

for $p \geq 1$. The p-affine perimeter of K is bounded by the area via the p-affine isoperimetric inequality:

$$\frac{\Omega_p^{2+p}(K)}{A^{2-p}(K)} \leq 2^{2+p}\pi^{2p},$$

with equality obtained if and only if for origin centered ellipses, [13]. If $p = 1$, this last inequality is know as affine isoperimetric inequality. We call the quantity $\frac{\Omega_p^{2+p}(K)}{A^{2-p}(K)}$, the p-affine isoperimetric ratio.

Let K be a convex body having origin in its interior. The dual convex body associated to K with respect to the origin, denoted by K°, is defined by

$$K^\circ = \{y \in \mathbb{R}^2 \mid x \cdot y \leq 1, \ \forall x \in K\}.$$

The area of K°, denoted by $A^\circ = A(K^\circ)$ can also be represented in terms of affine invariant quantities:

$$A^\circ = \frac{1}{2}\int_{\partial K} \frac{1}{\sigma^2} d\mathfrak{s} = \frac{1}{2}\int_{\mathbb{S}^1} \frac{1}{s^2} d\theta.$$

By the Blaschke-Santaló inequality we can bound the volume product:

$$A(K)A^\circ(K) \leq \pi^2,$$

with equality obtained if and only if for origin centered ellipses [14].

Let us outline our argument presented in the rest of this paper. Harnack's estimate is an important ingredient in our argument: We first convert Corollary 2.4 stated in the Gauss parametrization ($\mathcal{G}$) to a Corollary stated in the Euclidean parametrization ($\mathcal{E}$). By using the new Corollary, the evolution equation of the affine support function, monotonicity of the l-affine isoperimetric ratio, for $l \geq 2$, we obtain the asymptotic value of the affine support function as t approaches negative infinity. Then, using a standard ODE comparison theorem (Lemma 4.2, [8]) we prove that the volume product converges to its maximum value, which is achieved only for ellipses, as t converges to negative infinity. We proved in [8] that the volume product converges to its maximum value as t converges to the extinction time T. On the other hand, the the volume product is a monotone quantity (Proposition 2.2, [16]). Therefore, the the volume product must be constant along the p-flow which in turn implies that the only origin-symmetric, ancient solutions are ellipses. To carry out the outlined strategy we resort to affine differential geometry.

Let $K_0 \in \mathcal{K}_{sym}$. Consider a family $\{K_t\}_t \in \mathcal{K}_{sym}$, and their associated smooth embeddings $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$, which are evolving according to (1.1). Then, up to a time-dependant diffeomorphism, $\{K_t\}_t$ evolves according to

$$\frac{\partial}{\partial t} x := \sigma^{1-\frac{3p}{p+2}} \mathfrak{n}, \ x(.,0) = x_{K_0}(\cdot), \ x(\cdot, t) = x_{K_t}(\cdot). \tag{3.1}$$

Therefore, classification of compact, origin-symmetric ancient solutions to (1.1) is equivalent to the classification of compact, origin-symmetric ancient solutions to (3.1). In what follows our reference flow is the evolution equation (3.1).

Note that as a family of convex bodies evolve according to the evolution equation (3.1), then in the Gauss parametrization their support functions and curvatures evolve, according to Lemma 2.1. Consequently, as $\{K_t\}_t$ evolve according to the evolution equation (3.1) we have, in the Gauss parametrization, that $\partial_t \left(\frac{\kappa^{1/3}}{s}\right) \geq 0$, by Corollary 2.4.

It can be easily verified that the evolution equation of a geometric quantity Q in the Euclidean parametrization and in the Gauss parametrization along the flow (3.1) are related by

$$(\partial_t Q)_{\mathcal{G}} = (\partial_t Q)_{\mathcal{E}} - Q_{\mathfrak{s}}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}},$$

see Lemma 2.3, [7]. In particular, we have

$$\begin{aligned} 0 \geq (\partial_t \sigma)_{\mathcal{G}} &= (\partial_t \sigma)_{\mathcal{E}} - \sigma_{\mathfrak{s}}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}} \\ &= (\partial_t \sigma)_{\mathcal{E}} + \left(\frac{3p}{p+2}-1\right)\sigma_{\mathfrak{s}}^2\sigma^{-\frac{3p}{p+2}}. \end{aligned}$$

Therefore, the affine support function is non-increasing along the flow (3.1) in the Euclidean parametrization. Furthermore, we have proved

Corollary 3.1. *Every ancient solution to the flow (1.1) satisfies:*

$$(\partial_t \sigma)_{\mathcal{E}} \leq -\left(\frac{3p}{p+2}-1\right)\sigma_{\mathfrak{s}}^2\sigma^{-\frac{3p}{p+2}}.$$

For the rest of this paper we work in the Euclidean parametrization and for simplicity we drop the subscript $\mathcal{E}$. The next two Lemmas were proved in [8].

Lemma 3.2. [8] *Let $\gamma_t := \partial K_t$ be the boundary of a convex body K_t evolving under the flow (3.1). Then the following evolution equations hold:*

(1) $\frac{\partial}{\partial t}\sigma = \sigma^{1-\frac{3p}{p+2}}\left(-\frac{4}{3}+\left(\frac{p}{p+2}+1\right)\left(1-\frac{3p}{p+2}\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma}+\frac{p}{p+2}\sigma_{\mathfrak{s}\mathfrak{s}}\right)$,

(2) $\frac{d}{dt}A = -\Omega_p$.

Lemma 3.3. [8] *The following evolution equation for Ω_l under the p-flow for every $l \geq 2$ and $p \geq 1$ holds:*

$$\frac{d}{dt}\Omega_l(t) = \frac{2(l-2)}{l+2}\int_{\gamma_t}\sigma^{1-\frac{3p}{p+2}-\frac{3l}{l+2}}d\mathfrak{s} + \frac{18pl}{(l+2)^2(p+2)}\int_{\gamma_t}\sigma^{-\frac{3p}{p+2}-\frac{3l}{l+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}. \tag{3.2}$$

Let us denote $\max\limits_{\gamma_t}\sigma$ and $\min\limits_{\gamma_t}\sigma$ by σ_M and σ_m, respectively.

Lemma 3.4. *There is a constant c, depending on p, such that*

$$\frac{\sigma_M}{\sigma_m} \leq c$$

along the p-flow on the time interval $(-\infty, T)$.

Proof. By Corollary 3.1 and part (1) of Lemma 3.2 we have

$$\begin{aligned} -\left(\frac{3p}{p+2}-1\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma^2} &\geq \frac{\partial_t\sigma}{\sigma^{2-\frac{3p}{p+2}}} \\ &= -\frac{4}{3\sigma}+\left(\frac{p}{p+2}+1\right)\left(1-\frac{3p}{p+2}\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma^2}+\frac{p}{p+2}\frac{\sigma_{\mathfrak{s}\mathfrak{s}}}{\sigma}. \end{aligned} \tag{3.3}$$

Integrating the inequality (3.3) against $d\mathfrak{s}$ we obtain

$$\begin{aligned} \frac{4}{3}\int_{\gamma_t}\frac{1}{\sigma}d\mathfrak{s} &\geq \frac{p}{p+2}\left(2-\frac{3p}{p+2}\right)\int_{\gamma_t}\frac{\sigma_{\mathfrak{s}}^2}{\sigma^2}d\mathfrak{s} \\ &= \frac{p}{p+2}\left(2-\frac{3p}{p+2}\right)\int_{\gamma_t}(\ln(\sigma))_{\mathfrak{s}}^2 d\mathfrak{s} \end{aligned} \tag{3.4}$$

Set $d_p := \frac{p}{p+2}\left(2-\frac{3p}{p+2}\right)$. Thus, d_p is positive if

$$1 \leq p < 4.$$

Applying the Hölder inequality to the left-hand side and right-hand side of inequality (3.4) we get

$$\frac{\left(\int |(\ln \sigma)_{\mathfrak{s}}|d\mathfrak{s}\right)^2}{\Omega_1} \leq d_p' A^{\circ\ \frac{1}{2}}\Omega_1^{\frac{1}{2}} = d_p' \frac{A^{\frac{1}{2}}A^{\circ\ \frac{1}{2}}\Omega_1^{\frac{1}{2}}}{A^{\frac{1}{2}}},$$

for a new positive constant d_p'. Here we used the identities $\displaystyle\int_{\gamma_t}\frac{1}{\sigma^2}d\mathfrak{s} = 2A^{\circ}$ and $\displaystyle\int_{\gamma_t} d\mathfrak{s} = \Omega_1$. Now using the Blaschke-Santaló inequality, $AA^{\circ} \leq \pi^2$, we have

$$\left(\ln\frac{\sigma_M}{\sigma_m}\right)^2 \leq d_p''\left(\frac{\Omega_1^3}{A}\right)^{\frac{1}{2}},$$

for a new constant d_p''. Observe that the affine isoperimetric ratio, $\frac{\Omega_1^3}{A}$, is bounded by the affine isoperimetric inequality. Therefore we find that

$$\frac{\sigma_M}{\sigma_m} \leq c \tag{3.5}$$

on the time interval $(-\infty, T)$, for some positive universal constant c, depending only on p. □

Let $\{K_t\}_t$ be a solution of the flow (3.1). Then, the family of convex bodies, $\{\tilde{K}_t\}_t$, defined by

$$\tilde{K}_t := \sqrt{\frac{\pi}{A(K_t)}}K_t$$

is called a normalized solution to the p-flow, equivalently a solution that the area is fixed and is equal to π.

We denote every quantity associated to the normalized solution with an over-tilde. For example, the support function, curvature and the affine support function of $\tilde{K}$ is denoted by $\tilde{s}$, $\tilde{\kappa}$ and $\tilde{\sigma}$, respectively.

Lemma 3.5. *There is a constant c, depending on p, such that*

$$\frac{\tilde{\sigma}_M}{\tilde{\sigma}_m} \leq c. \tag{3.6}$$

for every normalized solution to the p-flow on the time interval $(-\infty, T)$.

Proof. The estimate (3.5) is scaling invariant. Therefore, the same estimate holds for the normalized solution. □

For the rest of this section we always assume that $l \geq 2$.

Lemma 3.6. *Along the p-flow we have*

$$\frac{d}{dt}\Omega_l(t) \geq \frac{l-2}{l+2}\frac{\Omega_l\Omega_p}{A} + \frac{18pl}{(l+2)^2(p+2)}\int_{\gamma_t}\sigma^{-\frac{3p}{p+2}-\frac{3l}{l+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}.$$

Proof. Before presenting a proof of the claim let us to state the following generalized Hölder inequality. If M is a compact manifold with a volume form $d\omega$, g is a continues function on M and F is decreasing real, positive function, then

$$\frac{\int_M gF(g)d\omega}{\int_M F(g)d\omega} \leq \frac{\int_M gd\omega}{\int_M d\omega}.$$

If F is strictly decreasing, then equality occurs if and only if g is constant.
Define $d\omega=\sigma d\mathfrak{s}$, $g=\sigma$ and $F(x):=x^{-\frac{3l}{l+2}}$. Furthermore, recall that for a convex body K in $\mathbb{R}^2$ we have $2A=\int_{\partial K}\sigma d\mathfrak{s}$. This implies

$$\int_{\partial K}\sigma^{1-\frac{3p}{p+2}-\frac{3l}{l+2}}d\mathfrak{s}\geq\frac{\Omega_l\Omega_p}{2A}.$$

The claim now follows by this last inequality and the evolution equation (3.2). □

Lemma 3.7. *Reciprocal of the l-affine isoperimetric ratio, $\frac{A^{2-l}(t)}{\Omega_l^{2+l}(t)}$, is non-decreasing along the p-flow.*

Proof.

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right)^{-1}\\
&=-\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right)^{-2}\frac{d}{dt}\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right)\\
&=-\left(\frac{\Omega_l^{l+2}(t)}{A^{2-l}(t)}\right)^{-2}\left(\frac{(2+l)\Omega_l^{l+1}(t)A^{2-l}(t)\frac{d}{dt}\Omega_l+(2-l)A^{1-l}(t)\Omega_l^{2+l}(t)\Omega_p(t)}{A^{2(2-l)}(t)}\right)\\
&=-\frac{\Omega_p(t)}{A(t)}\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\left((2+l)\frac{d}{dt}\Omega_l-(l-2)\frac{\Omega_l(t)\Omega_p(t)}{A(t)}\right)\\
&\leq-\frac{18pl}{(l+2)(p+2)}\frac{\Omega_p(t)}{A(t)}\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t}\sigma^{-\frac{3p}{p+2}-\frac{3l}{l+2}}\sigma_{\mathfrak{s}}^2d\mathfrak{s},
\end{aligned}$$

where we used Lemma 3.6 on the last inequality. □

In the rest of this paper we always assume that $1\leq p<4$.

Corollary 3.8. *There exists a constant $b_{l,p}>0$ depending on l and p such that*

$$\frac{A^{2-l}(t)}{\Omega_l^{2+l}(t)}<b_{l,p}$$

on $(-\infty,T)$.

Proof. Note that

$$\frac{A^{2-l}(t)}{\Omega_l^{2+l}(t)}=\frac{\left(\frac{1}{2}\int_{\partial\gamma_t}\sigma d\mathfrak{s}\right)^{2-l}}{\left(\int_{\partial\gamma_t}\sigma^{1-\frac{3l}{l+2}}d\mathfrak{s}\right)^{2+l}}$$

is a $GL(2)$ invariant quantity. Therefore, we need to only prove the claim after applying appropriate special linear transformations to the normalized solution of the p-flow. By the estimate (3.6) and the facts that $\max_{\mathbb{S}^1}\tilde{\sigma}\geq1$ and $\min_{\mathbb{S}^1}\tilde{\sigma}\leq1$ (see Lemma 2.4 in [10]) we have

$$\frac{1}{c}\leq\tilde{\sigma}=\frac{\tilde{s}}{\tilde{\kappa}^{\frac{1}{3}}}\leq c. \tag{3.7}$$

Observe that $\tilde{\sigma}$ is an $SL(2)$ invariant quantity. Therefore, the previous estimate holds even after applying an arbitrary special linear transformation. After applying length minimizing special linear transformation at each time t to the normalized solution of the p-flow, by John's lemma, the

support functions have uniform lower and upper bound on the time interval $(-\infty, T)$. Therefore, by the inequalities (3.7) curvature is uniformly bounded from below and above on the time interval $(-\infty, T)$. Now the claim follows as $d\mathfrak{s} = \mathfrak{r}^{2/3} d\theta$. $\square$

By the computation carried out in the proof of Lemma 3.7 we find

$$\frac{d}{dt}\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right)^{-1} \leq c_{l,p}\frac{d}{dt}\ln(A(t))\left[\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t}\left(\sigma^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}\right],$$

where $c_{l,p} := \dfrac{18pl}{(l+2)(p+2)\left(1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}\right)^2}$. Since $\frac{d}{dt}A(t) = -\Omega_p(t)$. This inequality will be used in the proof of the next Corollary.

Corollary 3.9. *If K_t evolves by (3.1), the following limit holds:*

$$\liminf_{t\to-\infty}\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t}\left(\sigma^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} = 0. \tag{3.8}$$

Proof. Suppose on the contrary. There exists an $\varepsilon > 0$ small such that

$$\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t}\left(\sigma^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} \geq \frac{\varepsilon}{c_{l,p}}$$

in a neighborhood of $-\infty$, on $(-\infty, -N)$ for large enough N. Then

$$\frac{d}{dt}\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right)^{-1} \leq \varepsilon\frac{d}{dt}\ln(A(t)).$$

Thus by Corollary 3.8

$$\begin{aligned}\left(\frac{\Omega_l^{2+l}}{A^{2-l}}\right)^{-1}(t) &\leq \left(\frac{\Omega_l^{2+l}}{A^{2-l}}\right)^{-1}(t_1) + \varepsilon\ln(A(t)) - \varepsilon\ln(A(t_1))\\ &\leq b_{l,p} + \varepsilon\ln(A(t)) - \varepsilon\ln(A(t_1)).\end{aligned}$$

Contradiction by letting t_1 approaches to $-\infty$: Since $\lim\limits_{t\to-\infty} A(t_1) = +\infty$, the right hand side becomes negative. $\square$

Corollary 3.10. *There is a sequence of time $\{t_k\}_{k\in\mathbb{N}}$ such that as t_k converges to $-\infty$ we have*

$$\lim_{t_k\to-\infty}\tilde{\sigma}(t_k) = 1.$$

Proof. Note that the quantity $\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t}\left(\sigma^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}$ is scaling invariant and $\frac{\tilde{A}^{3-l}(t)}{\tilde{\Omega}_p(t)\tilde{\Omega}_l^{l+3}(t)}$ is bounded from below by the l-affine isoperimetric inequality and by the p-affine isoperimetric inequality. Hence, Corollary 3.9 implies that there exists a sequence of time $\{t_k\}_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} t_k = -\infty$ and

$$\lim_{t_k\to-\infty}\int_{\tilde{\gamma}_{t_k}}\left(\tilde{\sigma}^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}} = 0.$$

On the other hand, by the Hölder inequality

$$\frac{\left(\tilde{\sigma}_M^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}-\tilde{\sigma}_m^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)^2}{\tilde{\Omega}_1}\leq\int_{\tilde{\gamma}_{t_k}}\left(\tilde{\sigma}^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)^2_{\tilde{\mathfrak{s}}}d\tilde{\mathfrak{s}}.$$

Therefore, by boundedness of $\tilde{\Omega}_1$ from above, by the affine isoperimetric inequality, we find that

$$\lim_{t_k\to-\infty}\left(\tilde{\sigma}_M^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}-\tilde{\sigma}_m^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)^2=0.$$

Since $\tilde{\sigma}_m\leq 1$ and $\tilde{\sigma}_M\geq 1$ the claim follows. $\square$

Lemma 3.11 (Monotonicity of the volume product)**.** [16] *The volume product, $A(t)A^\circ(t)$, is strictly increasing along the p-flow unless K_t is an ellipse centered at the origin.*

Corollary 3.12. *For any compact ancient solution to the p-flow we have*

$$\lim_{t\to-\infty}A(t)A^\circ(t)=\pi^2.$$

Proof. We first show that

$$\lim_{t_k\to-\infty}A(t)A^\circ(t)=\pi^2.$$

This is the consequence of Corollary 3.10 and a standard ODE comparison theorem (Lemma 4.2, [8]):

By Corollary 3.10, we have

$$\lim_{t_k\to-\infty}\tilde{\sigma}(t_k)=1. \tag{3.9}$$

Thus, by Lemma 4.2 in [8], there exist two families of origin-centered ellipses $\{\mathcal{E}_{in}(t_k)\}$, $\{\mathcal{E}_{out}(t_k)\}$ such that

$$\mathcal{E}_{in}(t_k)\subseteq\tilde{K}_{t_k}\subseteq\mathcal{E}_{out}(t_k) \tag{3.10}$$

and

$$\lim_{t_k\to-\infty}\sigma(\mathcal{E}_{in}(t_k))=\lim_{t_k\to-\infty}\sigma(\mathcal{E}_{out}(t_k))=1.$$

Evidently we can find an appropriate family of special linear transformations $\{L_{t_k}\}_{t_k}$ such that $L_{t_k}(\mathcal{E}_{out}(t_k))$ is a circle at each time t_k. Each such area preserving linear transformation L_{t_k} minimizes the Euclidean length of the ellipse $\mathcal{E}_{out}(t_k)$ at time t_k. Thus, the construction of $\mathcal{E}_{out}(t_k)$ and $\mathcal{E}_{in}(t_k)$ implies

$$\lim_{t_k\to-\infty}L_{t_k}(\mathcal{E}_{out}(t_k))=\lim_{t_k\to-\infty}L_{t_k}(\mathcal{E}_{in}(t_k))=\mathbb{S}^1$$

in the Hausdorff metric:

Since σ is invariant under $SL(2)$, we have $\sigma(L_{t_k}(\mathcal{E}_{out}(t_k)))=\sigma(\mathcal{E}_{out}(t_k)))$, therefore we have $\lim_{t_k\to-\infty}\sigma(L_{t_k}(\mathcal{E}_{out}(t_k)))=1$. This implies $\lim_{t_k\to-\infty}L_{t_k}(\mathcal{E}_{out}(t_k))=\mathbb{S}^1$ in the Hausdorff metric. Similarly, we have $\lim_{t_k\to-\infty}\sigma(L_{t_k}(\mathcal{E}_{in}(t_k))))=1$. This implies that

$$\lim_{t_k\to-\infty}A(L_{t_k}(\mathcal{E}_{in}(t_k)))=\pi.$$

As $L_{t_k}(\mathcal{E}_{in}(t_k))\subseteq L_{t_k}(\mathcal{E}_{out}(t_k))$, we conclude that $\lim_{t_k\to-\infty}L_{t_k}(\mathcal{E}_{in}(t_k))=\mathbb{S}^1$ in the Hausdorff metric.

Now, applying $\{L_{t_k}\}_{t_k}$ to the inclusions (3.10), we obtain that $L_{t_k}(\tilde{K}_{t_k})$ converges to the unit disk in the Hausdorff metric and therefore,

$$\lim_{t_k\to-\infty}A(t)A^\circ(t)=\pi^2.$$

Now monotonicity of the volume product, $A(t)A^\circ(t)$, stated in Lemma 3.11, finishes the proof. $\square$

4. Proof of the main Theorem

We now have gathered all the necessary ingredients to prove our main Theorem.

Theorem (Ancient solutions). *Let $1 \leq p < 4$. The only compact, origin-symmetric, ancient solutions to the p-flow are homothetic ellipses.*

Proof. We have proved in [8] that $\lim_{t_k \to T} A(t)A^{\circ}(t) = \pi^2$. On the other hand, by Corollary 3.12 we have $\lim_{t_k \to -\infty} A(t)A^{\circ}(t) = \pi^2$. Therefore, $A(t)A^{\circ}(t)$ achieves the same value at both ends of the interval $(-\infty, T)$. Since $A(t)A^{\circ}(t)$ is a monotone quantity we conclude that the volume product is constant on $(-\infty, T)$. Now Lemma 3.11 implies that K_t must be an ellipse centered at the origin of the plane for all $t \in (-\infty, T)$. $\square$

References

1. B. Andrews, *Contraction of convex hypersurfaces by their affine normal*, J. Diff. Geom. **43**, 207–230, 1996.
2. B. Andrews, *Harnack inequalities for evolving hypersurfaces*, Math. Zeitschrift **217**, 179–197, 1994.
3. B. Andrews, *Motion of hypersurfaces by Gauss curvature*, Pacific J. Math. **195**, No.1, 2000.
4. B. Chow, *On a formula of Daskalopoulos, Hamilton and Sesum*, `arXiv:1209.1784v1`.
5. P. Daskalopoulos, R. Hamilton, N. Sesum, *Classification of compact ancient solutions to the curve shortening flow*, J. Diff. Geom. **84**, 455–464, 2010.
6. P. Daskalopoulos, R. Hamilton and N. Sesum, *Classification of compact ancient solutions to the Ricci flow on surfaces*, `arXiv:0902.1158v2`.
7. M.N. Ivaki, *Affine-geometric Wirtinger inequality*, `arXiv:1205.6454v2`.
8. M.N. Ivaki, *Centro-affine curvature flows on centrally symmetric convex curves*, to appear in Trans. Amer. Math. Soc. `arXiv:1205.6456v2`.
9. M.N. Ivaki, *A flow approach to the L_{-2} Minkowski problem*, to appear in Adv. Appl. Math. `arXiv:1205.6455v2`.
10. M.N. Ivaki, *On stability of the p-affine isoperimetric inequality*, `arXiv:1210.0256v1`.
11. J. Loftin, M.P. Tsui, *Ancient solutions of the affine normal flow*, J. Diff. Geom. **78**, 113–162, 2008.
12. M. Ludwig and M. Reitzner, *A classification of $SL(n)$ invariant valuations*, Ann. of Math. **172**, 1223–1271, 2010.
13. E. Lutwak, The Brunn-Minkowski-Fiery theory II: Affine and geominimal surface areas, Adv. Math. **118**, 244–294, 1996.
14. L.A. Santaló, *An affine invariant for convex bodies of n-dimensional space*, Portugalia Math. **8**, 155–161, 1949.
15. G. Sapiro and A. Tannenbaum, *On affine plane curve evolution*, J. Funct. Anal. **119**, 79–120, 1994.
16. A. Stancu, *Centro-affine invariants for smooth convex bodies*, Int. Math. Res. Not. doi: 10.1093/imrn/rnr110, 2011.

Department of Mathematics and Statistics, Concordia University, Montreal, QC, Canada, H3G 1M8

E-mail address: `mivaki@mathstat.concordia.ca`